On the Non-Emptiness of the Core of a Cooperative Fuzzy Game

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Introduction

Consider first a cooperative game of n players.

The set of the players: $N = \{1, 2, \dots, n\}$

A **coalition** is any subset $K \subseteq N$ of the set N of the players.

- $K = \emptyset$ — empty coalition
- $K = N$ — grand coalition

The potency set of N: $\mathcal{P}(N) = \{ K : K \subseteq N \}$

— a collection of all coalitions that potentially can form

Introduction

The **coalition function** is any mapping

$$v: \mathcal{P}(N) \longrightarrow \mathbb{R}$$

$$v: \quad K \quad \longmapsto v(K)$$

such that

$$v(\emptyset) = 0$$

The number $v(K)$ is the worth of the coalition $K \subseteq N$,

i.e. the payoff which the coalition K will achieve if it is formed.

Introduction

The **coalition structure** is any partition of the set N of the players; that is, the coalition structure is any collection

$$\mathcal{S} = \{S_1, S_2, \dots, S_r\}$$

of coalitions such that

$$\bigcup_{p=1}^{r} S_p = N$$

and

$$\emptyset \notin \mathcal{S}$$

and

$$S_p \cap S_q = \emptyset \quad \text{if} \quad p \neq q \qquad \text{for} \quad p, q = 1, 2, \dots, r$$

Introduction

Let the players $1, 2, \ldots, n$ form a coalition structure

$$\mathcal{S} = \{S_1, S_2, \ldots, S_r\}$$

It follows that the coalitions $S_1, S_2, \ldots, S_r$ have formed, they exist, and their payoffs will be $v(S_1), v(S_2), \ldots, v(S_r)$, respectively.

Now the question is how the players of a coalition $S_1, S_2, \ldots, S_r$ will divide their respective payoff $v(S_1), v(S_2), \ldots, v(S_r)$ among themselves.

Introduction

The division of the payoff among the players is described by the payoff vector.

The **payoff vector** is any vector

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$$

where a_i is the payoff of the i-th player for $i = 1, 2, \ldots, n$.

Introduction

The **core** of the cooperative game, with transferable utility, given by the coalition function $v\colon \mathcal{P}(N) \to \mathbb{R}$ with respect to the coalition structure $\mathcal{S} = \{S_1, S_2, \dots, S_r\}$ is

$$\mathcal{C} = \left\{ \boldsymbol{a} \in \mathbb{R}^n : \begin{array}{ll} \sum_{i \in S} a_i = v(S) & \text{for} \quad S \in \mathcal{S} \\ \sum_{i \in K} a_i \geq v(K) & \text{for} \quad K \in \mathcal{P}(N) \setminus \mathcal{S} \end{array} \right\}$$

Introduction

In other words, the **core** is the set of the payoff vectors $\boldsymbol{a} \in \mathbb{R}^n$ that satisfy the condition of

- feasibility $\sum_{i \in S} a_i \leq v(S)$ for $S \in \mathcal{S}$
- efficiency $\sum_{i \in S} a_i \geq v(S)$ for $S \in \mathcal{S}$
- coalitional rationality $\sum_{i \in K} a_i \geq v(K)$ for $K \in \mathcal{P}(N) \setminus \mathcal{S}$

The question:

Is the core non-empty?

The Bondareva-Shapley Theorem

The core $\mathcal{C} = \{ \boldsymbol{a} \in \mathbb{R}^n : \sum_{i=1}^{n} a_i = v(N)$ and $\sum_{i \in K} a_i \geq v(K)$ for $K \in \mathcal{P}(N) \setminus \{N\} \}$ with respect to the coalition structure $\mathcal{S} = \{N\}$ is non-empty if and only if the game is balanced.

We should like to consider a general coalition structure $\mathcal{S} = \{S_1, S_2, \dots, S_r\}$ however.

Notice that the core is non-empty if and only if the following system of linear inequalities has a solution:

$$\sum_{i \in S} a_i \leq v(S) \qquad \text{for} \quad S \in \mathcal{S}$$

$$-\sum_{i \in K} a_i \leq -v(K) \qquad \text{for} \quad K \in \mathcal{P}(N)$$

Gale's Theorem of the alternative

Let $\boldsymbol{A}$ be a real matrix of type $m \times n$ and let $\boldsymbol{b}$ be a column vector of m real numbers. Then the system of linear inequalities

$$\boldsymbol{Ax} \leq \boldsymbol{b}$$

has no solution (where $\boldsymbol{x} \in \mathbb{R}^n$ is variable) if and only if

$$\exists \boldsymbol{\lambda}^{\mathrm{T}} \in \mathbb{R}^{1\times m},\ \ \boldsymbol{\lambda}^{\mathrm{T}} \geq \boldsymbol{0}^{\mathrm{T}}\!:\ \ \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{0}^{\mathrm{T}}\ \ \text{and}\ \ \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{b} < 0$$

In other words, the system $\boldsymbol{Ax} \leq \boldsymbol{b}$ has a solution if and only if

$$\forall \boldsymbol{\lambda}^{\mathrm{T}} \in \mathbb{R}^{1\times m},\ \ \boldsymbol{\lambda}^{\mathrm{T}} \geq \boldsymbol{0}^{\mathrm{T}}\!:\ \ \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{0}^{\mathrm{T}}\ \ \Longrightarrow\ \ \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{b} \geq 0$$

Characteristic vector of a coalition

Let the characteristic vector of a coalition $K \subseteq N$ be

$$\boldsymbol{\chi}^K = (\ \chi_1^K \quad \chi_2^K \quad \dots \quad \chi_n^K\ )$$

with

$$\chi_i^K = \begin{cases} 1 & \text{if} \quad i \in K \\ 0 & \text{if} \quad i \notin K \end{cases} \qquad \text{for} \quad i = 1, 2, \dots, n$$

Then the core is non-empty if and only if the following system of linear inequalities has a solution:

$$\boldsymbol{\chi}^S \boldsymbol{a} \leq v(S) \qquad \text{for} \quad S \in \mathcal{S}$$

$$\boldsymbol{\chi}^K \boldsymbol{a} \geq v(K) \qquad \text{for} \quad K \in \mathcal{P}(N)$$

where $\boldsymbol{a} \in \mathbb{R}^n$ is variable.

Balanced collection of coalitions

We say that a collection $\mathcal{T} = \{T_1, T_2, \dots, T_s\}$ of coalitions is **balanced** with respect to the coalition structure $\mathcal{S} = \{S_1, S_2, \dots, S_r\}$ if and only if

$$\sum_{q=1}^{s} \lambda_q \boldsymbol{\chi}^{T_q} = \sum_{p=1}^{r} \mu_p \boldsymbol{\chi}^{S_p}$$

for some balancing weights $\lambda_1, \lambda_2, \dots, \lambda_s \geq 0$ and

for some $\mu_1, \mu_2, \dots, \mu_r \geq 0$ such that $\mu_1 + \mu_2 + \cdots + \mu_r = 1$.

Balanced game

We say that the given *game* is **balanced**

with respect to the coalition structure $\mathcal{S} = \{S_1, S_2, \ldots, S_r\}$ if and only if

$$\sum_{q=1}^{s} \lambda_q v(T_q) \leq \sum_{p=1}^{r} \mu_p v(S_p)$$

for every balanced collection $\mathcal{T} = \{T_1, T_2, \ldots, T_s\}$ of coalitions.

The Bondareva-Shapley Theorem: a generalization

The core

$$\mathcal{C} = \left\{ \boldsymbol{a} \in \mathbb{R}^n : \begin{array}{ll} \sum_{i \in S} a_i \leq v(S) & \text{for} \quad S \in \mathcal{S} \\ \sum_{i \in K} a_i \geq v(K) & \text{for} \quad K \in \mathcal{P}(N) \end{array} \right\}$$

of the cooperative game, with transferable utility, given by the coalition function $v: \mathcal{P}(N) \to \mathbb{R}$ with respect to the coalition structure $\mathcal{S} = \{S_1, S_2, \dots, S_r\}$ is non-empty if and only if the game is balanced.

The core and balancedness
of fuzzy TU-games

Introduction to fuzzy TU-games

We consider a cooperative game of n players.

The set of the players: $N = \{1, 2, \dots, n\}$

A **coalition** is any fuzzy subset $\widetilde{K} \mathrel{\widetilde{\subseteq}} N$ of the set N of the players.
It is given by its **membership vector** $\boldsymbol{\kappa} \in [0,1]^N$

$$\boldsymbol{\kappa} = (\; \kappa_1 \quad \kappa_2 \quad \dots \quad \kappa_n \;)$$

with

$$\kappa_i \in [0,1] \qquad \text{for} \quad i = 1, 2, \dots, n$$

- empty coalition — $\chi^{\emptyset} = (\; 0 \quad 0 \quad \dots \quad 0 \;)$
- grand coalition — $\chi^{N} = (\; 1 \quad 1 \quad \dots \quad 1 \;)$

Introduction to fuzzy TU-games

The **fuzzy potency set** of N:

$$\tilde{\mathcal{P}}(N) = \left\{ \widetilde{K} : \widetilde{K} \mathrel{\tilde{\subseteq}} N \right\}$$

— a collection of all fuzzy coalitions
that potentially can form

— it is identified with the aforementioned set

$$[0,1]^N$$

of all the membership vectors

$$\boldsymbol{\kappa} \in [0,1]^N$$

Introduction to fuzzy TU-games

The **fuzzy coalition structure** is any indexed collection

$$\tilde{S} = \left(\tilde{S}_p\right)_{p \in \mathcal{R}}$$

of fuzzy coalitions $\tilde{S}_p \,\tilde{\subseteq}\, N$, where $\mathcal{R}$ is an index set, with membership vectors

$$\boldsymbol{\sigma}_p \in [0,1]^N \qquad \text{for} \quad p \in \mathcal{R}$$

such that

$$\sum_{p \in \mathcal{R}} \boldsymbol{\sigma}_p = \boldsymbol{\chi}^N \qquad \text{and} \qquad \boldsymbol{\sigma}_p \neq \boldsymbol{\chi}^{\emptyset} \qquad \text{for} \quad p \in \mathcal{R}$$

Remarks:

- The fuzzy coalition structure can be infinite (even though N is finite).
- Some fuzzy coalition $\tilde{S} \,\tilde{\subseteq}\, N$ can be present several times in the $\tilde{S}$.

Introduction to fuzzy TU-games

Let the players $1, 2, \dots, n$ form a fuzzy coalition structure

$$\tilde{\mathcal{S}} = \left(\tilde{S}_p\right)_{p \in \mathcal{R}}$$

It follows that the fuzzy coalitions $\tilde{S}_p$, for $p \in \mathcal{R}$, have formed and exist.

We have

$$0 \le \left(\boldsymbol{\sigma}_p\right)_i \le 1 \qquad \text{for} \quad p \in \mathcal{R} \qquad \text{and for} \quad i \in N$$

<u>Interpretation:</u>

- $\left(\boldsymbol{\sigma}_p\right)_i = 0$ — player i is not involved in coalition $\tilde{S}_p$ at all
- $\left(\boldsymbol{\sigma}_p\right)_i = 1$ — player i is involved in coalition $\tilde{S}_p$ for “full-time job”
- $0 < \left(\boldsymbol{\sigma}_p\right)_i < 1$ — player i is involved in coalition $\tilde{S}_p$ for “part-time job”

Introduction to fuzzy TU-games

Moreover, given the coalition structure $\tilde{\mathcal{S}} = \left(\tilde{S}_p\right)_{p\in\mathcal{R}}$, it may happen that

$$\tilde{S}_p = \tilde{S}_q \qquad \text{for some} \quad p, q \in \mathcal{R} \quad \text{with} \quad p \neq q$$

We interpret this fact so that the coalitions $\tilde{S}_p$ and $\tilde{S}_q$ are actually distinct and they endeavour in different branches of industry.

It follows that the total payoffs of the distinct coalitions $\tilde{S}_p$ and $\tilde{S}_q$, both of which exist at the same time, may be distinct too in general.

A new model of cooperative fuzzy TU-game

The cooperative fuzzy TU-game is given by the following pair of functions:

$$V : \mathcal{R} \to \mathbb{R}$$

and

$$v : [0,1]^N \to \mathbb{R}$$

with

$$v\left(\chi^{\emptyset}\right) = 0$$

The value $V(p)$ is the worth of the coalition $\tilde{S}_p$,

i.e. the payoff which the coalition $\tilde{S}_p$ receives,

i.e. the amount of some transferable utility assigned to the coalition $\tilde{S}_p$.

A new model of cooperative fuzzy TU-game

Assume that a new fuzzy coalition $\widetilde{K} \widetilde{\subseteq} N$ takes the opportunity to form, i.e. it forms, leaves the present coalition structure $\tilde{\mathcal{S}}$, and starts to endeavour in a new branch of industry.

We use the second function v now:

Let $\boldsymbol{\kappa} \in [0,1]^N$ be the membership vector of the fuzzy coalition $\widetilde{K} \widetilde{\subseteq} N$.

The number $v(\boldsymbol{\kappa})$ is the worth of the fuzzy coalition $\widetilde{K} \widetilde{\subseteq} N$, i.e. the payoff which the fuzzy coalition $\widetilde{K}$ will receive if it is formed.

The payoff matrix

The purpose is again that the players within each fuzzy coalition $\tilde{S}_p$, for $p \in \mathcal{R}$, divide the total payoff $V(p)$ of their coalition among themselves.

The division of the payoff among the players is now described by the **payoff matrix**

$$\boldsymbol{A} \in \mathbb{R}^{N \times \mathcal{R}}$$

where

$$a_{ip}$$

is the payoff of player i in coalition $\tilde{S}_p$ for $i \in N$ and for $p \in \mathcal{R}$.

The total payoff of player $i \in N$ is

$$\pi_i = \sum_{p \in \mathcal{R}} a_{ip}$$

The core

We define the **core** of the cooperative fuzzy TU-game given by its fuzzy coalition structure $\tilde{S} = \left(\tilde{S}_p\right)_{p \in \mathcal{R}}$, the function $V\colon \mathcal{R} \to \mathbb{R}$ of a coalition of this fuzzy coalition structure, and the fuzzy coalition function $v\colon [0,1]^N \to \mathbb{R}$ with $v\left(\chi^\emptyset\right) = 0$ to be the set

$$\mathcal{C} = \Big\{ \boldsymbol{A} \in \mathbb{R}^{N \times \mathcal{R}} \quad : \quad \left(\boldsymbol{\sigma}_p\right)_i = 0 \implies a_{ip} = 0 \qquad \text{for } i \in N \qquad \text{and for} \quad p \in \mathcal{R}$$

$$\sum_{i \in N} a_{ip} = V(p) \qquad \text{for} \quad p \in \mathcal{R}$$

$$\sum_{p \in \mathcal{K}} \sum_{i \in N} a_{ip} \geq v(\boldsymbol{\kappa}) \qquad \text{for} \quad \boldsymbol{\kappa} \in [0,1]^N \qquad \text{and}$$

$$\text{for finite } \mathcal{K} \subseteq \mathcal{R} \text{ such that} \quad \textstyle\sum_{p \in \mathcal{K}} \boldsymbol{\sigma}_p \geq \boldsymbol{\kappa} \Big\}$$

Balanced collection of fuzzy coalitions

We say that a *collection* $\left(\widetilde{K}_1, \widetilde{K}_2, \dots, \widetilde{K}_s\right)$ *of fuzzy coalitions* $\widetilde{K}_1, \widetilde{K}_2, \dots, \widetilde{K}_s \widetilde{\subseteq} N$, with membership vectors $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \dots, \boldsymbol{\kappa}_s \in [0,1]^N$, *along with* a *collection* $(\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_s)$ *of finite index sets* $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_s \subseteq \mathcal{R}$, such that $\sum_{p \in \mathcal{K}_q} \boldsymbol{\sigma}_p \geq \boldsymbol{\kappa}_q$ for $q = 1, 2, \dots, s$,

<u>is **balanced**</u> with respect to the fuzzy coalition structure $\tilde{\mathcal{S}} = \left(\tilde{S}_p\right)_{p \in \mathcal{R}}$ if and only if

$$\sum_{q=1}^{s} \sum_{p \in \mathcal{K}_q} \lambda_q \left[\boldsymbol{\sigma}_p\right] = \sum_{\rho=1}^{r} \mu_{p_\rho} \left[\boldsymbol{\sigma}_{p_\rho}\right]$$

for some balancing weights $\lambda_1, \lambda_2, \dots, \lambda_s \geq 0$, for some natural number $r \in \mathbb{N}$, for some indices $p_1, p_2, \dots, p_r \in \mathcal{R}$, and for some $\mu_{p_1}, \mu_{p_2}, \dots, \mu_{p_r} \geq 0$ such that $\mu_{p_1} + \mu_{p_2} + \cdots + \mu_{p_r} = 1$.

Balanced cooperative fuzzy TU-game

We say that the given cooperative fuzzy *game* is **balanced**

with respect to the fuzzy coalition structure $\tilde{\mathcal{S}} = \left(\tilde{S}_p\right)_{p\in\mathcal{R}}$ if and only if

$$\sum_{q=1}^{s} \lambda_q v\left(\boldsymbol{\kappa}_q\right) \leq \sum_{\rho=1}^{r} \mu_{p_\rho} V\left(p_\rho\right)$$

for every balanced collection $\left(\widetilde{K}_1, \widetilde{K}_2, \dots, \widetilde{K}_s\right)$ of fuzzy coalitions

along with the corresponding collection $(\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_s)$ of the finite index sets.

The Bondareva-Shapley Theorem: a generalization II

The core

$$\mathcal{C} = \Big\{ \boldsymbol{A} \in \mathbb{R}^{N \times \mathcal{R}} \ : \quad (\boldsymbol{\sigma}_p)_i = 0 \implies a_{ip} = 0 \quad \text{for } i \in N \quad \text{and for} \quad p \in \mathcal{R}$$

$$\sum_{i \in N} a_{ip} = V(p) \quad \text{for} \quad p \in \mathcal{R}$$

$$\sum_{p \in \mathcal{K}} \sum_{i \in N} a_{ip} \geq v(\boldsymbol{\kappa}) \quad \text{for} \quad \boldsymbol{\kappa} \in [0,1]^N \quad \text{and}$$

$$\text{for finite } \mathcal{K} \subseteq \mathcal{R} \text{ such that} \quad \textstyle\sum_{p \in \mathcal{K}} \boldsymbol{\sigma}_p \geq \boldsymbol{\kappa} \ \Big\}$$

of the cooperative fuzzy TU-game is non-empty if and only if the game is balanced.